Topological currents for arbitrary chiral groups in three space dimensions

1977 J. Phys. A: Math. Gen. 10 1397

(http://iopscience.iop.org/0305-4470/10/8/018)

J. Phys. A: Math. Gen., Vol. 10, No. 8, 1977. Printed in Great Britain. © 1977

Topological currents for arbitrary chiral groups in three space dimensions

C J Isham
Blackett Laboratory, Physics Department, Imperial College, London SW7 2BZ, UK

Received 7 April 1977

Abstract. It is shown that the homotopy classes of soliton-type solutions to chiral field theories in three space dimensions are determined entirely by cohomological properties. This result is employed to construct topological currents which are identically conserved and whose integrated time component contains all the homotopical information.

1. Introduction

An important ingredient in much of the recent work on solitons and instantons is the classification of solutions to field equations in terms of homotopy classes. In the present paper we are concerned with situations where the third homotopy group of a differentiable manifold M is involved. This may be regarded as the group of homotopy classes of smooth maps from the three-sphere $\mathscr{S}^3$ into M. There are at least three cases where this arises.

(i) Consider solutions to the classical Yang–Mills equations in four dimensions ('Euclideanised' space–time). If these solutions are required to possess a total action which is finite then the fields $F_{\mu\nu}$ must vanish at large distances. More precisely if a three-sphere of radius R is constructed around the origin of coordinates, we require the field $F_{\mu\nu}$ to tend to zero faster than $R^{-3/2}$. One way of ensuring this is to force the Yang–Mills potential A_μ to be pure gauge at the three-sphere boundary of four-dimensional Euclidean space. Thus on this boundary

$$A_\mu = g^{-1}(x)\partial_\mu g(x) \tag{1.1}$$

for some function g mapping this sphere into the gauge group G. Such functions can be classified into homotopy classes, with G playing the role of the differential manifold M (Belavin *et al* 1975).

(ii) Consider again solutions to the Yang–Mills equations but concentrate now on *spatially* localised gauge transformations. Such transformations are generated by functions g from physical three-space R^3 into the gauge group, which become the identity group element e at spatial infinity. This means that the whole two-sphere boundary of R^3 is identified to a single point and mapped into e and such maps are precisely equivalent to functions from a three-sphere into G (Jackiw and Rebbi 1976).

(iii) Finally suppose we are interested in classical solutions to theories involving non-linear realisations of some compact group $\mathscr{K}$ acting on a manifold M. Such theories were popular ten years ago with $\mathscr{K}$ being a chiral group such as SU(2)×SU(2) or

SU(3)×SU(3) and have been re-investigated recently from a soliton viewpoint (Honerkamp *et al* 1976, Duff and Isham 1976, 1977, Deser *et al* 1976, Barnes *et al* 1977, Nicole 1977). The manifold M is parametrised by a set of fields $\phi^1 \ldots \phi^n$ ($n = \dim M$) which possess a non-linear transformation under $\mathscr{K}$ induced by the group action on M. The Lagrangian is typically

$$\mathscr{L} = \tfrac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial_\mu \phi^j \tag{1.2}$$

with g_{ij} being a $\mathscr{K}$-invariant metric on M. If finite energy solutions to the static field equations are required then ϕ^i must become a constant at spatial infinity. Thus once again the boundary of R^3 is identified to a point and the possibility arises of classifying solutions according to the homotopy class of maps from the three-sphere into M. Such a situation can also arise with Higgs–Kibble fields in Yang–Mills gauge theory, the manifold M labelling the different possible vacuum states (Coleman 1975).

Clearly an essential requirement for the use of these homotopical concepts is the ability to actually compute the homotopy class to which the function of interest belongs. Most of the previous work on this problem has been in the situation where M is itself a sphere. Typically a two-sphere $\mathscr{S}^2$ for models in two space dimensions and a three-sphere $\mathscr{S}^3$ for theories in three space dimensions. (Maps from $\mathscr{S}^3$ into $\mathscr{S}^2$ have also been investigated (Hertel 1977).) Such maps may be labelled by their degree (an integer) and the theory is well developed (Arafune *et al* 1975). However for general manifolds M the problem is more complicated and forms the subject matter of this paper. I am mainly concerned with case (iii) above (the non-linear chiral models) although many of the results are of general applicability. If $\mathscr{K}$ is a chiral group, $G\times G$ say, then M is normally chosen to be the coset space $G\times G/G_\Delta$ where G_Δ is the diagonal subgroup of $G\times G$. This quotient space is diffeomorphic to G so we are really concerned with constructively classifying elements of the third homotopy group $\Pi_3(G)$ of the Lie group G. It is a standard result that for most simple groups $\Pi_3(G)\simeq \mathbb{Z}$ (the integers, see § 4 for details) and hence in analogue with previous results the aim is to find 'topological current' $\mathscr{N}^\mu$ on four-dimensional space–time which is identically conserved (i.e. independent of any equation of motion), chirally invariant and with the property that the integral of $\mathscr{N}^0$ over any space-like surface is the integer labelling the homotopy class. The conservation of $\mathscr{N}^\mu$ and the identification of the boundary of R^3 to a point guarantees that this number is independent of the space-like hypersurface chosen, even for time-dependent solutions to the field equations. It is also desirable to construct lower bounds on the energy of a static solution in terms of this topological 'charge'. Such bounds are considered in § 3 and can provide a powerful proof of the stability under small disturbances. This is an essential property if quantum corrections are to be considered (Duff and Isham 1977).

The crucial observation is that within the context of differential geometry the quantities (like $\mathscr{N}^0$) that can be integrated are differential forms. This in turn suggests that it may be profitable to recast the problem in terms of cohomology theory. This is done in § 2 where an explicit algorithm for constructing the desired topological currents is presented. In general it is easier to construct cohomology groups than homotopy groups and correspondingly in many respects the former carry less information than the latter. An essential feature of the present situation is that for maps into a Lie group G the cohomological information is the *same* as that obtained from homotopy theory. Strictly speaking this result is necessary to fully justify the use of the topological currents constructed in § 2. However the derivation is rather technical, using a string of known topological properties of Lie groups, and is deferred until the final section of the paper.

2. Topological currents for an arbitrary chiral group

Consider a continuous map f from $\mathscr{S}^3$ into the compact simple Lie group G. This induces homomorphisms from the real (singular) homology and cohomology groups (Greenberg 1967, Spanier 1966):

$$f^*: \qquad H^n(G;R)\to H^n(\mathscr{S}^3;R) \qquad n=0,1,2,\ldots. \tag{2.1}$$

$$f_*: \qquad H_n(\mathscr{S}^3;R)\to H_n(G;R) \tag{2.2}$$

Let f and h be two such maps. Then a basic theorem of algebraic topology states that if f and h are homotopic (denoted $f\sim h$) then $f_*=h_*$ and $f^*=h^*$. We are interested in the converse, in particular when does $f^*=h^*$ imply that $f\sim h$? If G were itself a three-sphere (i.e. if $G=\mathrm{SU}(2)$) then the Brouwer degree theorem (Spanier 1966) shows that this converse does hold and, as demonstrated in § 4, this remains true for many compact Lie groups G. Now $H^n(\mathscr{S}^3;R)=0$ unless $n=3$ when $H^3(\mathscr{S}^3;R)\approx R$ (using augmented groups so that $H^0(\mathscr{S}^3;R)\approx 0$). Thus in order to investigate the homotopic properties of a map f: $\mathscr{S}^3\to G$ it suffices to investigate the induced map f^* from $H^3(G;R)$ into $H^3(\mathscr{S}^3;R)\approx R$. Using De Rham's theorem (Goldberg 1962) $H^3(G;R)$ may be identified with the space of closed differential three-forms on the differentiable manifold G modulo exact three-forms. The general theory of Lie group cohomology tells us that (Goldberg 1962)

$$H^1(G;R)\approx H^2(G;R)\approx 0 \tag{2.3}$$

$$H^3(G;R)\approx R \tag{2.4}$$

and that the group $H^3(G;R)$ is generated by a three-form invariant under left and right multiplication of G by itself. To construct such a form recall that the Lie algebra of G may be identified with a set of left-invariant vector fields on G, X_a, $a=1\ldots\dim G$, satisfying the commutation relations

$$[X_a, X_b]=C_{ab}{}^c X_c \tag{2.5}$$

where $C_{ab}{}^c$ are a set of real structure constants. If $\{M^1\ldots M^m, m\equiv\dim G\}$ is a local coordinate system on G (physically they are ultimately the fields in the theory) then X_a has the components $\underset{a}{\xi}^i(M)$ i.e. $X_a=\underset{a}{\xi}^i(\partial/\partial M^i)$ and equation (2.5) implies

$$\underset{a}{\xi}^i\underset{b}{\xi}^j{}_{,i}-\underset{b}{\xi}^i\underset{a}{\xi}^j{}_{,i}=C_{ab}{}^c\underset{c}{\xi}^j \qquad a,b,c;i,j=1\ldots m. \tag{2.6}$$

In the theory of non-linear realisations of the chiral group $\mathscr{K}=G\times G$ the $\underset{a}{\xi}^i$ appear as coefficients in the 'infinitesimal' transformation δM^i with left-chiral group parameters $\psi^1\ldots\psi^m$:

$$\delta M^i=\underset{a}{\xi}^i(M)\psi^a. \tag{2.7}$$

Let $\{w^a\}$ denote the set of one-forms dual to $\{X_a\}$. In components

$$w^a=\overset{a}{\xi}_i\,\mathrm{d}M^i \tag{2.8}$$

where

$$\overset{a}{\xi}_i\underset{b}{\xi}^i=\delta_b{}^a, \qquad \overset{a}{\xi}_i\underset{a}{\xi}^j=\delta_i^j. \tag{2.9}$$

These forms satisfy the Cartan–Maurer structure equations (which are equivalent to (2.5)):

$$dw^a = -\tfrac{1}{2}C^a{}_{bc}\, w^b \wedge w^c \tag{2.10}$$

where $\wedge$ and d denote the exterior product and derivative respectively. Now consider the three-form

$$\begin{aligned}\tau &= C_{abc} w^a \wedge w^b \wedge w^c \\ &\equiv C_{abc} \overset{a}{\xi}_i \overset{b}{\xi}_j \overset{c}{\xi}_k \, dM^i \wedge dM^j \wedge dM^k \qquad (2.11)\\ &= -2 w_a \wedge dw^a \qquad \text{(from (2.10))}\\ &\equiv -2 \overset{a}{\xi}_i \underset{a}{\xi}_{j,k}\, dM^i \wedge dM^j \wedge dM^k. \qquad (2.12)\end{aligned}$$

(Note that it is assumed that the basis $\{X_a\}$ for the Lie algebra of G has been chosen so that the Cartan–Killing form is simply the Kronecker delta. This is being used to raise and lower the indices a, b, c.)

Clearly τ is a G-invariant form, it is closed ($d\tau = 0$) by virtue of the Jacobi identities and in fact (Goldberg 1962) it generates $H^3(G; R)$. Now suppose that $f\colon \mathscr{S}^3 \to G$ is a solution to the static field equations. Then $f^*\tau$ is a three-form on $\mathscr{S}^3$ given explicitly by

$$f^*\tau = C_{abc} \overset{a}{\xi}_i \overset{b}{\xi}_j \overset{c}{\xi}_k \frac{\partial M^i(\boldsymbol{x})}{\partial x^r} \frac{\partial M^j(\boldsymbol{x})}{\partial x^s} \frac{\partial M^k(\boldsymbol{x})}{\partial x^t}\, dx^r \wedge dx^s \wedge dx^t \tag{2.13}$$

where $\{x^r\}$, $r = 1, 2, 3$, are a set of coordinates on R^3 (with boundary identification) and $M^i(\boldsymbol{x})$ means the functions $M^i \circ f\colon \mathscr{S}^3 \to R$. If the 'left-chiral currents' J_{ar} are defined by

$$J_{ar} \equiv \underset{a}{\xi}_i \frac{\partial M^i}{\partial x^r} \qquad a, i = 1 \ldots m;\ r = 1, 2, 3 \tag{2.14}$$

then

$$f^*\tau = C_{abc} J^a{}_r J^b{}_s J^c{}_t \, dx^r \wedge dx^s \wedge dx^t. \tag{2.15}$$

For the simple Lagrangian in equation (1.2) the invariant metric is

$$g_{ij} = \overset{a}{\xi}_i \overset{a}{\xi}_j \tag{2.16}$$

and J_{ar} really *are* the spatial components of the Noether currents associated with the left-chiral group invariance. However as shown in Deser *et al* (1976) and Fadeev (1977) a simple Derrick-theorem-type of argument implies that the Lagrangian in (1.2) generates no non-trivial, finite energy, static solutions in three space dimensions. There are more complicated higher-derivative Lagrangians which do lead to such solutions but J_{ar} will no longer be the associated Noether currents.

We may summarise the results so far by saying that two solutions f, h to the static field equations will be homotopic if and only if the three-forms $f^*\tau$ and $h^*\tau$ are cohomologous. This in turn is true if and only if the three-forms differ by a divergence, which may be easily checked by integration. Thus f and h are homotopic if and only if

$$\int_{\mathscr{S}^3} C^{abc} J_{ar} J_{bs} J_{ct} \epsilon^{rst}\, d^3\boldsymbol{x} = \int_{\mathscr{S}^3} C^{abc} J'_{ar} J'_{bs} J'_{ct} \epsilon^{rst}\, d^3\boldsymbol{x} \tag{2.17}$$

where J_{ar} and J'_{ar} are the currents produced by f and h respectively.

A topological current associated with this construction may be obtained in the following way (Fadeev 1977). Let ψ: $\mathcal{S}^3 \times R \to G$ be a time-dependent function. Then $\psi^*\tau$ is a three-form on $\mathcal{S}^3 \times R$ and

$$\psi^*\tau = C_{abc} J^a{}_\alpha J^b{}_\beta J^c{}_\gamma \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \tag{2.18}$$

with

$$J^a{}_\alpha = \xi^a_i \frac{\partial M^i}{\partial x^\alpha} \qquad \alpha = 0, 1, 2, 3. \tag{2.19}$$

Let $\mathcal{N}$ denote the one-form dual to $\psi^*\tau$:

$$\mathcal{N} = * \, \psi^*\tau = \mathcal{N}_\mu \, dx^\mu \tag{2.20}$$

where

$$\mathcal{N}_\mu = C_{abc} J^a{}_\alpha J^b{}_\beta J^c{}_\gamma \epsilon^{\alpha\beta\gamma}{}_\mu. \tag{2.21}$$

Then $d\tau = 0$ implies that

$$\partial_\mu \mathcal{N}^\mu = 0 \tag{2.22}$$

and

$$\int_{\mathcal{S}^3} f^*\tau = \int_{\mathcal{S}^3} \mathcal{N}^0 \, d^3\boldsymbol{x}. \tag{2.23}$$

Since in addition $\mathcal{N}_\mu$ is G-invariant this is evidently the topological current being sought. One anticipates that there will be some renormalised version of $\mathcal{N}^0$ with the property that $\mathcal{N}^0 \, d^3\boldsymbol{x}$ is actually an integer, presumably related to the corresponding member of the third homotopy group. This is discussed in § 4.

3. Energy bounds

Energy bounds on classical static solutions of the form

$$\text{energy} > |\text{topological charge}| \tag{3.1}$$

are very useful. In particular if a classical solution saturates (3.1) (i.e. the equality holds) then for any sufficiently small perturbation the energy can only increase. This is because such a perturbation cannot change the topological charge (which is assumed to be an integer—§ 4) and the result implies in particular that the static solution really is at a minimum rather than a maximum of the potential. Without this topological tool such stability can only be verified by solving the small-disturbance equation—a task which in practice can be extremely difficult (Duff and Isham 1977).

Bounds of the type in (3.1) can obviously be obtained by spatially integrating inequalities of the form

$$\text{energy density} > |\mathcal{N}^0|. \tag{3.2}$$

To proceed any further we clearly need to know what energy density is being employed or, equivalently, the form of the Lagrangian. The most general chiral invariant Lagrangian can be expressed as an algebraic function of the currents $J_{\alpha\mu}$ with the space–time indices being saturated with the Minkowski metric or $\epsilon^{\mu\nu\alpha\beta}$ and the internal labels by the Kronecker delta, structure constants or other covariants (such as d^{abc} in SU(3)). Typical examples might be

$$\mathscr{L} = J_{a\mu} J^{a\mu} \tag{3.3}$$

or

$$\mathscr{L} = C^{abc} C_a^{de} J_{b\mu} J_d^{\mu} J_{c\nu} J_e^{\nu}. \tag{3.4}$$

The Lagrangian in equation (1.2) is in fact proportional to (3.3) whilst (3.4) contains four powers of field derivatives. As mentioned already, something like (3.4) is necessary in three space dimensions if stable finite energy solutions are to exist. Since

$$\mathscr{N}^0 = C_{abc} J_r^a J_s^b J_t^c \epsilon^{rst} \tag{3.5}$$

one obvious useful inequality is (Fadeev 1977)

$$(J_{ar} - \lambda C_a^{bc} J_{bs} J_{ct} \epsilon_r^{st})^2 \geqslant 0 \tag{3.6}$$

where the square of the bracket means summing over $a = 1, \ldots, m$ and $r = 1, 2, 3$ and where λ is any real number. Expanding (3.6) gives

$$J_{ar} J^{ar} + 2\lambda^2 C_a^{bc} C^{ade} J_{bs} J_d^s J_{ct} J_e^t \geqslant 2\lambda \mathscr{N}^0. \tag{3.7}$$

Either sign for λ may be chosen and its modulus varied at will either to optimise the bound or to make the left-hand side equal a specific Hamiltonian density.

Another possibility is

$$(J_{ar} J_{bs} - \mu C^c{}_{ab} \epsilon^t{}_{rs} J_{ct})^2 \geqslant 0 \tag{3.8}$$

which implies

$$(J_{ar} J^{ar})^2 + 4\mu^2 J_{ar} J^{ar} \geqslant 2\mu \mathscr{N}^0 \tag{3.9}$$

where again μ can be freely chosen. Linear combinations of (3.7) and (3.9) may also be employed if this is appropriate for the Lagrangian under consideration.

Finally

$$(J_{ar} J_{bs} J_{ct} - \epsilon C_{abc} \epsilon_{rst})^2 \geqslant 0 \tag{3.10}$$

which implies

$$(J_{ar} J^{ar})^3 \geqslant -12 m \epsilon^2 + 2\epsilon \mathscr{N}^0. \tag{3.11}$$

It is assumed throughout that the structure constants have been normalised so that $C_a{}^{bc} C^d{}_{bc} = 2\delta_a^d$.

As it stands the left-hand side of (3.11) involves six derivatives. The bound is unlikely to be useful in this form although it might conceivably be helpful in the form

$$(J_{ar} J^{ar})^{3/2} \geqslant (-12 m \epsilon^2 + 2\epsilon \mathscr{N}^0)^{1/2} \tag{3.12}$$

since, as was pointed out in Deser *et al* (1976), the somewhat exotic Lagrangian

$$\mathscr{L} = (g_{ij}\, \partial_\mu M^i\, \partial_\mu M^j)^{3/2} = (J_{a\mu} J^{a\mu})^{3/2} \tag{3.13}$$

avoids the Derrick scaling arguments in three space dimensions. (Of course the integral of the square root on the right-hand side is not equal to the square root of the integral!)

For any given group one can attempt to find solutions to the field equations which saturate these inequalities. In the case, for example, of (3.6) this involves finding a solution for which

$$J_{ar} = \lambda C_a^{\ bc} J_{bs} J_{ct} \epsilon_r^{\ st}. \tag{3.14}$$

This is reminiscent of the self-duality equations in the theory of Yang–Mills instantons. Equations (3.14) are *first*-order partial differential equations for $M^i(\boldsymbol{x})$ and are hence in principle easier to solve than the second-order equations of motion, which for arbitrary $\mathscr{L}(J)$ are

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial J_\mu^{\ a}} = 0. \tag{3.15}$$

However, unlike the analogous Yang–Mills case, it is by no means obvious that a solution of (3.14) will automatically satisfy (3.15) or indeed that there are any functions at all that satisfy both (3.14) and (3.15) simultaneously. This is a problem for further research.

4. Cohomological and homotopical equivalence

We now consider the problem of the equivalence between the cohomological and homotopical properties of a solution. It will be necessary in this section to be more precise in the use of mathematical language. Let E^3 denote the closed ball in $\boldsymbol{R}^3$; $E^3 = \{\boldsymbol{x} \in \boldsymbol{R}^3 | \boldsymbol{x} \cdot \boldsymbol{x} \leqslant 1\}$. This ball has a boundary $\mathscr{S}^2$ and its interior is a homeomorphic image of the space upon which the static solutions are defined. Such a solution f maps the boundary to some point $p_0 \in G$ and we write f as a map between pairs:

$$f: \qquad (E^3, \mathscr{S}^2) \to (G, p_0). \tag{4.1}$$

Consider integral relative homology and cohomology groups. Then f induces the homomorphisms

$$f^*: \qquad H^3(G, p_0; \mathbb{Z}) \to H^3(E^3, \mathscr{S}^2; \mathbb{Z}) \tag{4.2}$$

$$f_*: \qquad H_3(E^3, \mathscr{S}^2; \mathbb{Z}) \to H_3(G, p_0; \mathbb{Z}) \tag{4.3}$$

and the question of interest is: 'If f_1 and f_2 are two such maps with $f_1^* = f_2^*$, are f_1 and f_2 homotopic?' The answer is as follows.

Proposition 1. Let f_1 and f_2 be two continuous maps from the pair $(E^3, \mathscr{S}^2)$ into (G, p_0) where G is one of the compact, simply connected, classical simple Lie groups. Then $f_1^* = f_2^*$ implies that f_1 and f_2 are homotopic.

Proof. (i) Let ι be the canonical generator (Spannier 1966) of $H_3(E^3, \mathscr{S}^2; \mathbb{Z}) \simeq \mathbb{Z}$ and let $h := f_{1*} - f_{2*}$ and $k := f_1^* - f_2^*$. Denote the pairing between a cochain c and chain z by $\langle c, z \rangle$ and let $\alpha: H^3(G, p_0; \mathbb{Z}) \to H_3(G, p_0; \mathbb{Z})^*$ be the canonical homomorphism (Greenberg 1967) between the cohomology group and the dual of the homology group defined by

$$\alpha([c])([z]) = \langle c, z \rangle \qquad \text{for all } [c] \in H^3(G, p_0; \mathbb{Z}), [z] \in H_3(G, p_0; \mathbb{Z}).$$

Then for all $[c] \in H^3(G, p_0; \mathbb{Z})$, $\alpha([c])(h(\iota)) = \langle c, h(\iota) \rangle = \langle kc, \iota \rangle = 0$ since by assumption $k = f_1^* - f_2^* = 0$. However since $\mathbb{Z}$ is a principal ideal domain α is an epimorphism and therefore $h(\iota)$ is annihilated by all members of $H_3(G, p_0; \mathbb{Z})^*$. Thus $h(\iota)$ belongs to the torsion subgroup of $H_3(G, p_0; \mathbb{Z})$.

(ii) We would like to show that this torsion subgroup vanishes so that $h(\iota)=0$. The previously quoted results on Lie group cohomology involve *real* coefficients (being concerned with invariant differential forms) and tell us only that the rank of $H_3(G, p_0; \mathbb{Z})$ is one. We will prove the desired result using a method which has the virtue of introducing some results that will be necessary later.

The nth absolute homotopy group of G may be defined (Spanier 1966, Sze-Tsen Hu 1959) as the set of homotopy classes of maps from $(E^n, \mathscr{S}^{n-1})$ into (G, p_0) and denoted $\Pi_n(G, p_0)$. Strictly speaking there is a different group for each point $p_0 \in G$. However since G is a topological group there is a natural isomorphism (Steenrod 1951) between these different homotopy groups and we shall usually simply write $\Pi_n(G)$. (In particular f_1 and f_2 belong to the same class in $\Pi_3(G, p_0)$ if and only if they are freely homotopic.) Let f: $(E^3, \mathscr{S}^2) \to (G, p_0)$. Then there is a natural homomorphism (Spanier 1966, Sze-Tsen Hu 1959)

$$\phi: \Pi_3(G, p_0) \to H_3(G, p_0; \mathbb{Z})$$
$$[f] \rightsquigarrow f_*(\iota) \tag{4.4}$$

where $[f]$ denotes the homotopy class of f and ι is the canonical generator of $H_3(E^3, \mathscr{S}^2; \mathbb{Z})$ referred to above. Now since G is assumed simply connected, $\Pi_1(G) = 0$. Furthermore for any compact Lie group $\Pi_2(G) = 0$. The fundamental Hurewicz theorem (Spanier 1966) then states that ϕ is an *isomorphism*. The third homotopy groups of the classical simple Lie groups are well known (Husemoller 1966):

$$\Pi_3(\mathrm{SU}(n)) \approx \mathbb{Z} \qquad n \geqslant 2 \tag{4.5}$$
$$\Pi_3(\mathrm{Sp}(n)) \approx \mathbb{Z} \qquad n \geqslant 1 \tag{4.6}$$
$$\Pi_3(\mathrm{Spin}(n)) \approx \mathbb{Z} \qquad n \geqslant 5 \tag{4.7}$$

where Spin (n) is the universal covering group of $\mathrm{SO}(n)$, and hence in all these cases $H_3(G, p_0; \mathbb{Z})$ is isomorphic to the integers. (For the semi-simple group Spin(4), $\Pi_3(\mathrm{Spin}(4)) \approx \mathbb{Z} \oplus \mathbb{Z}$.)

(iii) In particular there is no torsion subgroup of $H_3(G, p_0; \mathbb{Z})$ and hence $h(\iota) = 0$. Under the homomorphism ϕ, $[f_1] - [f_2]$ is mapped onto $f_{1*}(\iota) - f_{2*}(\iota) = h(\iota) = 0$. Hence using the Hurewicz isomorphism theorem again it follows that $[f_1] = [f_2]$. Thus $f_1 \sim f_2$ which is the desired result.

The point p_0 plays no real role in the above result. Indeed as already indicated it can be omitted in the homotopy theory. Since $H^3(G, p_0; \mathbb{Z}) \approx H^3(G; \mathbb{Z})$ and $H_3(G, p_0; \mathbb{Z}) \approx H_3(G; \mathbb{Z})$ the statement in proposition 1 remains true if f_1 and f_2 map the boundary $\mathscr{S}^2$ into two different points and if f_i^* are regarded as homomorphisms between $H^3(G; \mathbb{Z})$ and $H^3(E^3, \mathscr{S}^2; \mathbb{Z})$.

Let us now return to the problem of normalising the topological current constructed in § 2.

Proposition 2. Using the notation of § 2 $\int_{\mathscr{S}^3} f^* \tau'$ is an integer M for some suitably normalised three-form $\tau' = r_0 \tau$, $r_0 \in R$.

Proof. f should properly be regarded as map from $(E^3, \mathscr{S}^2)$ into (G, p_0). There is a natural projection map p: $(E^3, \mathscr{S}^2) \to (\mathscr{S}^3, x_0)$ in which the boundary $\mathscr{S}^2$ of E^3 is identified to a point x_0. This induces isomorphisms between the various homology and cohomology groups and hence f can be regarded as generating a homomorphism (also denoted f^*) between $H^3(G; \mathbb{Z})$ and $H^3(\mathscr{S}^3; \mathbb{Z})$. The three-form τ is a generator of the group $H^3(G; R)$ which by virtue of the universal coefficient theorem (Greenberg 1967, Spanier 1966) is isomorphic to $R \otimes H^3(G; \mathbb{Z})$. Hence there exists some real number r_0 such that $r_0\tau$ is a generator for the free group $H^3(G; \mathbb{Z}) \approx \mathbb{Z}$. Now choose some orientation of $\mathscr{S}^3$ and let Γ be the associated generator of $H_3(\mathscr{S}^3; \mathbb{Z}) \approx \mathbb{Z}$. There will exist some generator $e \in H^3(\mathscr{S}^3; \mathbb{Z}) \approx \mathbb{Z}$ such that $(\alpha(e))(\Gamma) = 1$. Then $f^*(r_0\tau)$ must be some integral multiple of this generator: $f^*(r_0\tau) = Me$, $M \in \mathbb{Z}$. Now view these objects as elements of real cohomology groups. In the De Rham theorem the pairing between a cocycle and cycle is obtained by integrating the cocycle (now a closed differential form) over the cycle. Thus

$$\int_\Gamma f^*(r_0\tau) = M(\alpha(e))(\Gamma) = M \tag{4.8}$$

which, since integrating over Γ is equivalent to integrating over $\mathscr{S}^3$ with a specified orientation, is the required result.

The final problem is to fix r_0. If we could find a generator γ for $H_3(G; \mathbb{Z})$ this would be sufficient because using De Rham cohomology again

$$\int_\gamma r_0\tau = 1 \qquad \text{i.e.} \qquad r_0^{-1} = \int_\gamma \tau. \tag{4.9}$$

Since by the Hurewicz theorem $\Pi_3(G) \approx H_3(G, \mathbb{Z})$ it suffices to find a generator for $\Pi_3(G)$. Suppose $G = \mathrm{SU}(n)$ for some $n \geq 3$ and consider the following part of the homotopy exact sequence (Sze-Tsen Hu 1959, Steenrod 1951, Husemoller 1966) for the fibre bundle $\mathrm{SU}(n)$ over base space $\mathrm{SU}(n)/\mathrm{SU}(n-1)$ with fibre $\mathrm{SU}(n-1)$:

$$\to \Pi_4(\mathrm{SU}(n)/\mathrm{SU}(n-1)) \to \Pi_3(\mathrm{SU}(n-1)) \xrightarrow{i_*} \Pi_3(\mathrm{SU}(n)) \to \Pi_3(\mathrm{SU}(n)/\mathrm{SU}(n-1)) \to \ldots$$

where i denotes the subgroup embedding of $\mathrm{SU}(n-1)$ into $\mathrm{SU}(n)$ and i_* is the associated homomorphism of homotopy groups. The quotient space $\mathrm{SU}(n)/\mathrm{SU}(n-1)$ is diffeomorphic to $\mathscr{S}^{2n-1}$ and $\Pi_3(\mathscr{S}^{2n-1}) \approx \Pi_4(\mathscr{S}^{2n-1}) \approx 0$ for $n \geq 3$. Thus this portion of the exact sequence reduces to

$$0 \to \Pi_3(\mathrm{SU}(n-1)) \xrightarrow{i_*} \Pi_3(\mathrm{SU}(n)) \to 0$$

and hence i_* is an isomorphism. Some isomorphism is of course already implied in the tabulated forms of the third homotopy groups in (4.5), indeed this is partly how the results quoted in (4.5) are obtained. The crucial feature for our purposes is that this isomorphism is induced by the *subgroup embedding*. Suppose that $G = \mathrm{SU}(3)$. The group $\mathrm{SU}(2)$ is diffeomorphic to $\mathscr{S}^3$ and hence there is a canonical generator $\tilde{\Gamma}$ of $\Pi_3(\mathrm{SU}(2))$. Then by the result above $i_*\tilde{\Gamma}$ is a generator for $\Pi_3(\mathrm{SU}(3))$. Now the embedding of $\mathrm{SU}(3)$ in $\mathrm{SU}(4)$ leads to a generator of $\mathrm{SU}(4)$ and so on. Thus for any sequence $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \mathrm{SU}(4) \subset \ldots \mathrm{SU}(n)$ a generator of $\Pi_3(\mathrm{SU}(n))$ is obtained. In fact if j: $\mathrm{SU}(2) \to \mathrm{SU}(n)$ is the embedding of $\mathrm{SU}(2)$ into $\mathrm{SU}(n)$ then the homotopy class $[j]$ of j can serve as the generator of $\Pi_3(\mathrm{SU}(n)) \approx H_3(\mathrm{SU}(n); \mathbb{Z})$. Thus in practice r_0 can be determined from (4.8) by integrating the three-form τ over the $\mathrm{SU}(2)$ subgroup of $\mathrm{SU}(n)$. It is then clear that the integer M appearing in (4.8) is the same as that classifying the element of the homotopy group $\Pi_3(\mathrm{SU}(n))$.

Using a general property of topological groups (Steenrod 1951) the nth homotopy class $M[j]$ contains the function

$$j^{(M)}: \quad \begin{array}{c} SU(2) \to SU(n) \\ g \rightsquigarrow g^M \end{array}$$

where g^M really means $(j(g))^M$. Since $(j(g))^M$ lies in the SU(2) subgroup of SU(n) this means that any solution to the static field equations is homotopic to a function that lies purely in the chosen SU(2) subgroup! Unfortunately there is no reason why this function should itself solve the field equations so this procedure does not immediately reduce the SU(n) problem to the SU(2) problem (this is an interesting topic for further research).

These results can be readily generalised to the symplectic groups using the isomorphisms

$$\mathrm{Sp}(1) \approx \mathrm{SU}(2)$$

$$\mathrm{Sp}(n)/\mathrm{Sp}(n-1) \approx \mathscr{S}^{4n-1}.$$

Similarly for the spin groups

$$\mathrm{Spin}(3) \approx \mathrm{SU}(2)$$

$$\mathrm{Spin}(n)/\mathrm{Spin}(n-1) \approx \mathscr{S}^{n}$$

and by choosing some embedding of Spin(3) in Spin(5) the same procedure can be followed.

Finally we note that the restriction that G be simply connected may be removed. Any non-simply connected Lie group G is isomorphic to the quotient of its simply connected covering group U with some subgroup K of its centre. For the classical groups under discussion K is always a finite discreet group and the projection $p: U \to U/K$ induces an isomorphism between third homotopy groups (using the exact sequence for a bundle). The following map diagram is commutative:

$$\begin{array}{ccc} \Pi_3(U/K) & \overset{\phi}{\rightarrowtail} & H_3(U/K; \mathbb{Z}) \\ p_* \uparrow & & \uparrow p_* \\ \Pi_3(U) & \overset{\phi}{\rightarrowtail} & H_3(U; \mathbb{Z}) \end{array}$$

with the left-hand vertical and lower horizontal arrows being isomorphisms. The right-hand vertical map is not an isomorphism since $H_3(U/K; \mathbb{Z})$ may contain a torsional subgroup. It is however a monomorphism between $H_3(U; \mathbb{Z})$ and the free subgroup of $H_3(U/K; \mathbb{Z})$ and by commutativity the same is true of

$$\phi: \quad \Pi_3(U/K) \rightarrowtail H_3(U/K; \mathbb{Z}).$$

This is sufficient to complete the proof (part (iii)) of proposition 1 and the proof of proposition 2 can be adapted in a similar way.

Acknowledgments

I am grateful to D A Nicole for telling me about his exact solutions to the SU(3) chiral model prior to their publication. It was this information which inspired the present work.

References

Arafune J, Freund P G O and Goebal C J 1975 *J. Math. Phys.* **16** 435
Barnes K J, Ketley I J, Nicole D A and O'Donnell P J 1977 *J. Phys. G: Nucl. Phys.* submitted for publication
Belavin A A, Polyakov A M, Schwartz A S and Tyupkin Yu S 1975 *Phys. Lett.* **59B** 85
Coleman S 1975 *Erice Summer School Lectures* (New York: Academic)
Deser S, Duff M J and Isham C J 1976 *Nucl. Phys.* B **114** 29
Duff M J and Isham C J 1976 *Nucl. Phys.* B **108** 130
—— 1977 *Phys. Rev.* in the press
Fadeev L D 1977 *CERN Preprint* CERN TH. 2188
Goldberg S I 1962 *Curvature and Homology* (New York: Academic)
Greenberg M 1967 *Lectures on Algebraic Topology* (New York: Benjamin)
Hertel J 1977 *University of Hamburg Preprint*
Honerkamp J, Patani A, Schlindwein M and Shafi Q 1976 *Lett. Nuovo Cim.* **15** 97
Husemoller D 1966 *Fibre Bundles* (New York: McGraw-Hill)
Jackiw R and Rebbi C 1976 *Phys. Rev. Lett.* **37** 172
Nicole D A 1977 *J. Phys. G: Nucl. Phys.* in the press
Spanier E H 1966 *Algebraic Topology* (New York: McGraw-Hill)
Steenrod N 1951 *The Topology of Fibre Bundles* (Princeton: Princeton University Press)
Sze-Tsen Hu 1959 *Homotopy Theory* (New York: Academic)